A CHARACTERIZATION OF THE MOONSHINE VERTEX OPERATOR ALGEBRA BY MEANS OF VIRASORO FRAMES

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Abstract. In this article, we show that a framed vertex operator algebra V satisfying the conditions: (1) V is holomorphic (i.e, V is the only irreducible V-module); (2) V is of rank 24; and (3) $V_1 = 0$; is isomorphic to the moonshine vertex operator algebra $V^\natural$ constructed by Frenkel-Lepowsky-Meurman [12].

1. Introduction

The moonshine vertex operator algebra $V^\natural$ constructed by Frenkel-Lepowsky-Meurman [12] is one of the most important examples of vertex operator algebra (VOA). In the introduction of their book [12], Frenkel, Lepowsky and Meurman conjectured that $V^\natural$ can be characterized by the following three conditions:

(1) the VOA $V^\natural$ is the only irreducible ordinary module for itself;

(2) the rank (or central charge) of $V^\natural$ is 24;

(3) $V_1^\natural = 0$.

While conditions (2) and (3) are clear from the construction, condition (1) was proved by Dong [1] using the 48 mutually commuting Virasoro elements of central charge 1/2 discovered in [11]. The discovery of the 48 commuting Virasoro algebras inside $V^\natural$ also inspired the study of framed vertex operator algebras [2, 18]. In fact, a lot of progress have been made on the study of the moonshine VOA using the 48 commuting Virasoro algebras [2, 13, 14, 16, 19].

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In this article, we shall prove a weak version of FLM conjecture by adding an extra assumption that V contains 48 mutually commuting Virasoro elements of central charge 1/2 (i.e., V is a framed VOA).

Theorem 1.1. *Let V be a framed vertex operator algebra satisfying the conditions: (1) V is the only irreducible module for itself; (2) V is of rank 24; and (3) $V_1 = 0$. Then V is isomorphic to the moonshine vertex operator algebra $V^\natural$.*

FLM's construction [12] of the moonshine VOA $V^\natural$ is the first mathematical example of the so-called $\mathbb{Z}_2$-orbifold construction and it is closely related to the lattice VOA associated with the Leech lattice. Let Λ be the Leech lattice and V_Λ the lattice associated with Λ. As shown in [12], we can extend the -1 map on Λ to an automorphism θ of V_Λ by

$$\theta : \alpha_1(i_1)\cdots\alpha_k(i_k)e^{\alpha} \mapsto (-1)^k\alpha_1(i_1)\cdots\alpha_k(i_k)e^{-\alpha} \text{ for } \alpha_1, ..., \alpha_k, \alpha \in \Lambda. \tag{1.1}$$

Let V_Λ^T be the unique irreducible θ-twisted module. Then the moonshine VOA $V^\natural$ is given by

$$V^\natural = V_\Lambda^+ \oplus V_\Lambda^{T,+},$$

where V_Λ^+ and $V_\Lambda^{T,+}$ are the fixed point subspaces of θ on V_Λ and V_Λ^T, respectively. By the construction, there is a natural involution $t \in \mathrm{Aut}(V^\natural)$ such that $t|_{V_\Lambda^+} = 1$ and $t|_{V_\Lambda^{T,+}} = -1$. This automorphism t belongs to the $2B$ conjugacy class of the Monster group and the top weight of the unique irreducible t-twisted module over $V^\natural$ is 1 [13, 14]. By performing the t-orbifold construction on $V^\natural$, one can recover the Leech lattice VOA V_Λ [13, 14, 16].

Our main strategy is to reverse the above construction and try to obtain the Leech lattice VOA V_Λ by using the $\mathbb{Z}_2$-orbifold construction on V. The key point is that for any framed VOA V, one can easily define some involutions on V by using the frame structure. Such kind of involutions are often called τ-involutions or Miyamoto involutions (cf. [17]). If $V_1 = 0$, we can define an involution τ on V such that the top weight of the irreducible τ-twisted module V^T is 1. Then by performing τ-orbifold construction on V, we obtain a VOA

$$V(\tau) = V^{\langle\tau\rangle} \oplus (V^T)^{\langle\tau\rangle},$$

where $V^{\langle\tau\rangle}$ and $(V^T)^{\langle\tau\rangle}$ are the fixed point subspaces of τ in V and V^T respectively. We shall show that the weight one subspace $V(\tau)_1$ of $V(\tau)$ is nontrivial and the Lie algebra structure on $V(\tau)_1$ is abelian. Hence, by a result of Dong and Mason [9, Theorem 3], $V(\tau)$ is isomorphic to the Leech lattice VOA V_Λ. Therefore, by reversing the orbifold construction on $V(\tau)$, we obtain that $V^{\langle\tau\rangle} \cong V_\Lambda^+$ and $V \cong V_\Lambda^+ \oplus V_\Lambda^{T,+} \cong V^\natural$ as V_Λ^+-module. It is well-known that there exists a unique simple vertex operator algebra structure on $V_\Lambda^+ \oplus V_\Lambda^{T,+}$ (cf. [8, 13]) and thus we conclude that V and $V^\natural$ are isomorphic simple vertex operator algebras.

Another uniqueness result of the moonshine VOA $V^\natural$ is also obtained in [3] under a different set of assumptions. Again, the existence of 48 commuting Virasoro algebras of central charge 1/2 is crucial to their argument.

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2. Framed vertex operator algebra

First we shall recall the definition of framed vertex operator algebras and review several important results [2, 16, 18, 19, 20].

Definition 2.1. ([2, 19]) A simple vertex operator algebra (V, ω) is called *framed* if there exists a mutually orthogonal set $\{e^1, \dots, e^n\}$ of Virasoro elements of central charge 1/2 such that $\omega = e^1 + \cdots + e^n$ and each e^i generates a simple Virasoro vertex operator algebra isomorphic to $L(1/2, 0)$. The full sub VOA F generated by $e^1, \dots, e^n$ is called an *Virasoro frame* or simply a *frame* of V.

Let (V, ω) be a framed VOA with a frame F. We denote by $\mathrm{Vir}(e^i)$ the Virasoro subVOA generated by e^i. Then

$$F \cong \mathrm{Vir}(e^1) \otimes \cdots \otimes \mathrm{Vir}(e^i) \cong L(1/2, 0)^{\otimes n}.$$

Since the Virasoro VOA $L(1/2,0)$ is rational, V is a direct sum of irreducible F-submodules $\otimes_{i=1}^n L(1/2,h_i)$ with $h_i \in \{0,1/2,1/16\}$, namely

$$V = \bigoplus_{h_i\in\{0,1/2,1/16\}} m_{h_1,\dots,h_n} L(1/2,h_1)\otimes\cdots\otimes L(1/2,h_n),$$

where $m_{h_1,\dots,h_n}\in\mathbb{N}$ denotes the multiplicity.

For each irreducible F-module $\otimes_{i=1}^n L(1/2,h_i)$, we define its binary 1/16-*word* (or τ-*word*) $(\alpha_1,\cdots,\alpha_n)\in\mathbb{Z}_2^n$ by $\alpha_i=1$ if and only if $h_i=1/16$. For $\alpha\in\mathbb{Z}_2^n$, denote by V^α the sum of all irreducible F-submodules of V whose 1/16-words are equal to α. Define $D=\{\alpha\in\mathbb{Z}_2^n \mid V^\alpha\neq 0\}$. Then D is a linear code and we have the *1/16-word decomposition*

$$V=\bigoplus_{\alpha\in D} V^\alpha.$$

It is shown in Dong et al. [11] that

$$V^0 = \bigoplus_{h_i\in\{0,1/2\}} m_{h_1,\dots,h_n} L(1/2,h_1)\otimes\cdots\otimes L(1/2,h_n)$$

is a subalgebra of V and the multiplicity $m_{h_1,\cdots,h_n}\le 1$ for $h_i\in\{0,1/2\}$. Thus we obtain another linear code $C:=\{(2h_1,\cdots,2h_n)\in\mathbb{Z}_2^n \mid h_i\in\{0,1/2\},\ m_{h_1,\cdots,h_n}\neq 0\}$ and V^0 can be decomposed as

$$V^0 = \bigoplus_{\alpha=(\alpha_1,\cdots,\alpha_n)\in C} L(1/2,\alpha_1/2)\otimes\cdots\otimes L(1/2,\alpha_n/2). \tag{2.1}$$

The VOA V^0 is often called the code vertex operator algebra associated with the code C and denoted by M_C. Note that the VOA structure of V^0 is uniquely determined by the code C [2, 18].

Summarizing, there exists a pair of even linear codes (C,D) such that V is an D-graded extension of a code VOA M_C associated to C. We call the pair (C,D) the *structure codes* of a framed VOA V associated with the frame F. Since the powers of z in an $L(1/2,0)$-intertwining operator of type $L(1/2,1/2)\times L(1/2,1/2)\to L(1/2,1/16)$ are half-integral, the structure codes (C,D) satisfy $C\subset D^\perp$. Moreover, $C=D^\perp$ if and only if V is holomorphic (cf. [2, 16, 20]).

Let V be a framed VOA with the structure codes (C,D), i.e., $V=\oplus_{\alpha\in D}V^{\alpha}$, and $V^0=M_C$. For a binary codeword $\beta\in\mathbb{Z}_2^n$, we define a linear map $\tau_\beta:V\to V$ by

$$\tau_\beta=(-1)^{\langle\alpha,\beta\rangle}v,\qquad \text{for } v\in V^{\alpha}. \tag{2.2}$$

By the fusion rules, it is easy to show that τ_β is an automorphism of V (cf. [17]). This automorphism is often called a τ-involution (or Miyamoto involution). Let $P=\{\tau_\beta\mid\beta\in\mathbb{Z}_2^n\}$ be the subgroup generated by the τ-involutions. Then $P\cong\mathbb{Z}_2^n/D^{\perp}$ and $V^0=V^P$ is the fixed point subalgebra. Thus, all $V^{\alpha},\alpha\in D$, are irreducible modules over $V^0=M_C$ (cf. [6]).

Next we shall recall a very important result from [16].

Theorem 2.2 (Theorem 5.6 of [16])**.** *Let $V=\oplus_{\alpha\in D}V^{\alpha}$ be a framed VOA with structure codes (C,D). Then $V^{\alpha},\alpha\in D$, are all simple current modules over the code VOA $V^0=M_C$.*

The following two results follow immediately from the general arguments on simple current extensions [4, 15, 21].

Theorem 2.3 (Corollary 5.7 of [16])**.** *Let $V=\oplus_{\alpha\in D}V^{\alpha}$ be a framed VOA with structure codes (C,D). Let W be an irreducible V^0-module. Then there exists a unique $\eta\in\mathbb{Z}_2^n$ up to $D^{\perp}$ such that W can be uniquely extended to an irreducible τ_η-twisted V-module which is given by $V\boxtimes_{V^0}W$ as a V^0-module.*

Theorem 2.4 (cf. [14, 16])**.** *Let $V=\oplus_{\alpha\in D}V^{\alpha}$ be a holomorphic framed VOA with structure codes (C,D). For any $\delta\in\mathbb{Z}_2^n$, denote*

$$D^0=\{\alpha\in D\,|\,\langle\alpha,\delta\rangle=0\}\qquad\textit{and}\qquad D^1=\{\alpha\in D\,|\,\langle\alpha,\delta\rangle\neq 0\}.$$

Define

$$V(\tau_\delta)=\begin{cases}(\bigoplus_{\alpha\in D^0}V^{\alpha})\oplus(\bigoplus_{\alpha\in D^1}M_{\delta+C}\boxtimes_{M_C}V^{\alpha}) & \textit{if } \mathrm{wt}\,\delta \textit{ is odd,}\\ (\bigoplus_{\alpha\in D^0}V^{\alpha})\oplus(\bigoplus_{\alpha\in D^0}M_{\delta+C}\boxtimes_{M_C}V^{\alpha}) & \textit{if } \mathrm{wt}\,\delta \textit{ is even.}\end{cases}$$

Then $V(\tau_\delta)$ is also a holomorphic framed VOA. Moreover, the structure codes of $V(\tau_\delta)$ are given by (C,D) if $\mathrm{wt}\,\delta$ is odd and $(C\cup(\delta+C),D^0)$ if $\mathrm{wt}\,\delta$ is even.

Remark 2.5. The above construction of the holomorphic VOA $V(\tau_\delta)$ is referred to as a τ_δ-orbifold construction of V.

3. Strongly rational, holomorphic vertex operator algebra

In this section, we shall review some basic facts about strongly rational vertex operator algebra from [8, 9].

Definition 3.1. A VOA V is of *CFT-type* if the natural $\mathbb{Z}$-grading on V takes the form $V = V_0 \oplus V_1 \oplus \cdots$ with $V_0 = \mathbb{C}\mathbb{1}$.

Remark 3.2. Let $V = \oplus_{n=0}^{\infty} V_n$ be a VOA of CFT-type. Then the weight one subspace V_1 carries a structure of a Lie algebra with the bracket

$$[u, v] = u_0 v, \quad u, v \in V_1$$

and an invariant bilinear form defined by

$$\langle u, v\rangle \mathbb{1} = u_1 v \quad \text{ for } u, v \in V_1.$$

Definition 3.3 ([8])**.** A vertex operator algebra V is said to be *strongly rational* if it satisfies the following conditions:
1. V is of CFT type and $L(1)V_1 = 0$.
2. V is C_2-cofinite, i.e., $\dim V/C_2(V) < \infty$, where $C_2(V) = \mathrm{span}\{u_{-2}v \mid u, v \in V\}$.
3. V is rational.

Remark 3.4. All framed vertex operator algebras are strongly rational.

Recently, Dong and Mason [8, 9, 10] have a study of strongly rational, holomorphic vertex operator algebras of rank ≤ 24 by using the Lie algebra structure on V_1. Although their method is not very effective when $V_1 = 0$, they obtained the following characterization of the Leech lattice VOA V_Λ.

Theorem 3.5 (Dong-Mason [9])**.** *Let V be a strongly rational, holomorphic vertex operator algebra of rank 24 such that the Lie algebra on V_1 is abelian. Then $\dim V_1 = 24$ and V is isomorphic to the Leech lattice VOA V_Λ.*

4. Uniqueness of the moonshine vertex operator algebra

Let V be a holomorphic framed VOA of rank 24 such that $V_1 = 0$. We shall prove that V is isomorphic to the moonshine VOA $V^\natural$ by using the theorem of Dong-Mason (cf. Theorem 3.5). First let C and D be the structure codes of V. That means

$$V = \oplus_{\beta\in D} V^\beta, \quad V^0 = M_C \quad \text{and} \quad C = D^\perp.$$

Since $V_1 = 0$, the code C contains no element of weight 2. Now let $\delta = (110\dots0)$. Then $\delta \notin C = D^\perp$ and hence the automorphism τ_δ defines a (nontrivial) involution on V and the fixed point subVOA is given by

$$V^{\langle\tau_\delta\rangle} = \bigoplus_{\beta\in D^0} V^\beta, \qquad \text{where } D^0 = \{\beta \in D \mid \langle\beta,\delta\rangle = 0\}.$$

Now define the τ_δ-orbifold construction of V by

$$V(\tau_\delta) = \bigoplus_{\beta\in D^0} \left(V^\beta \oplus M_{\delta+C} \boxtimes_{M_C} V^\beta \right)$$

By Theorem 2.4, $V(\tau_\delta)$ is a holomorphic framed VOA of rank 24 and the structure codes of $V(\tau_\delta)$ is given by $(\tilde{C}, D^0)$, where $\tilde{C}$ is the binary code generated by C and δ.

Remark 4.1. One can define an involution g on $V(\tau_\delta)$ as follows:

$$g = \begin{cases} 1 & \text{on } \bigoplus_{\beta\in D^0} V^\beta, \\ -1 & \text{on } \bigoplus_{\beta\in D^0} M_{\delta+C} \boxtimes_{M_C} V^\beta. \end{cases}$$

Moreover, the fixed point subalgebra $V(\tau_\delta)^{\langle g\rangle} = V^{\langle\tau_\delta\rangle} = \bigoplus_{\beta\in D^0} V^\beta$.

Proposition 4.2. *$V(\tau_\delta)_1 \neq 0$ and the Lie algebra on $V(\tau_\delta)_1$ is abelian. As a consequence, $V(\tau_\delta)$ is isomorphic to the Leech lattice VOA V_Λ.*

Proof. Since $M_{\delta+C} \subset V(\tau_\delta)$ and $\mathrm{wt}\,\delta = 2$, we have $V(\tau_\delta)_1 \neq 0$. Now let g be the involution on $V(\tau_\delta)$ defined in Remark 4.1. Denote by $V(\tau_\delta)^+$ and $V(\tau_\delta)^-$ the fixed point subspace and the -1-eigenspace of g on $V(\tau_\delta)$, respectively. Since $V(\tau_\delta)^+ = V^{\langle\tau_\delta\rangle} \subset V$, the weight one subspace $V(\tau_\delta)^+_1$ of $V(\tau_\delta)$ is trivial and hence $V(\tau_\delta)_1 = V(\tau_\delta)^-_1$. Therefore, for any $u, v \in V(\tau_\delta)_1$, we have $u, v \in V(\tau_\delta)^-$ and thus

$$[u, v] = u_0 v \in [V(\tau_\delta)^-_1, V(\tau_\delta)^-_1] \subset V(\tau_\delta)^+_1 = 0.$$

Therefore, the Lie algebra $V(\tau_\delta)_1$ is abelian, and hence $V(\tau_\delta)$ is isomorphic to the Leech lattice VOA V_Λ by Theorem 3.5. ∎

Theorem 4.3. *Let V be a holomorphic framed VOA of rank 24 such that $V_1 = 0$. Then V is isomorphic to the moonshine VOA $V^\natural$.*

Proof. By the above proposition, we know that $V(\tau_\delta)$ is isomorphic to the Leech lattice VOA V_Λ. Moreover, g acts on $V(\tau_\delta)_1$ as -1 so that g is conjugate to the lift θ of -1 map on Λ (cf. Eq. 1.1) by Theorem D.6 of [2] (see also Lemma 7.15 of [16]). Therefore, we have $V(\tau_\delta)^{\langle g\rangle} = V^{\langle \tau_\delta\rangle} \cong V_\Lambda^+$. It is well-known (cf. [1, 13]) that V_Λ^+ has exactly 4 inequivalent irreducible modules, namely V_Λ^+, V_Λ^-, $V_\Lambda^{T,+}$ and $V_\Lambda^{T,-}$, and all these modules are simple currents. Their top weights are $0, 1, 2$ and $3/2$, respectively. Since V has integral weights and $V_1 = 0$, it is clear that

$$V \cong V_\Lambda^+ \oplus V_\Lambda^{T,+} \cong V^\natural$$

as V_Λ^+-modules. Then by the uniqueness of simple current extensions shown in [8], we can establish the desired isomorphism between V and $V^\natural$. ∎

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